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Integrating hedge algebras and optimization techniques to reduce forecasting errors in fuzzy time series model

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ABSTRACT

Accurate forecasting in fuzzy time series (FTS) models is essential for applications such as financial markets, traffic fatalities, and academic enrollments. However, a persistent challenge in FTS forecasting is determining the optimal interval lengths within the universe of discourse (UD), which significantly impacts prediction accuracy. This study presents a novel hybrid approach that combines Hedge Algebra (HA) with Particle Swarm Optimization (PSO) and Simulated Annealing (SA) to improve forecasting accuracy. HA enables adaptive, non-uniform interval partitioning based on linguistic semantics, while PSO and SA jointly refine these intervals to reduce forecasting errors. Unlike conventional FTS models with fixed partitioning, our approach leverages HA's mathematical structure alongside PSO's global search and SA's local refinement to enhance adaptability and robustness. The model is evaluated on diverse datasets, including enrollment data, traffic fatalities, and gasoline prices, demonstrating superior forecasting accuracy over existing FTS models, as measured by Mean Squared Error (MSE) and Root Mean Squared Error (RMSE).

1. Introduction

Over the past few years, researchers have developed FTS forecasting models using fuzzy set theory [1], [2] to tackle a wide range of real-world challenges, including academic enrollment trends, traffic accidents, stock market fluctuations, and annual population estimates. Unlike conventional statistical methods such as regression analysis, moving averages, autoregressive moving averages, and ARIMA models or individual machine learning models [3]–[7], these FTS models excelled in forecasting when historical data was linguistic or ambiguous [8], [9]. However, Song and Chissom's models faced challenges, including extensive computations with large fuzzy matrices and uncertainty in determining the interval lengths of the universe of discourse. To address these challenges, Chen [10] introduced a novel forecasting model based on fuzzy relation groups, which employed simple arithmetic operations in the defuzzification process. Other researchers emphasized the importance of assigning weights to resolve recurring fuzzy relationships and to reflect differences in their significance within the FTS model [11]–[13]. Effective approaches by Huarng [14] adjusted interval lengths to enhance forecasting accuracy. Subsequent developments led to high-order fuzzy time series models [15] and the two-factor FTS model [16]. Recent advancements have integrated various techniques at different stages of the FTS model, including automatic clustering methods [17], [18] and optimization techniques [19], [20], such as K-means clustering [21], [22] and C-means clustering [23]. In a departure from fuzzy approaches, hedge algebras [24] have emerged as a promising option for forecasting in FTS. Researchers have applied hedge algebra (HA) to model linguistic domains and variables, eliminating the need for data fuzzification and defuzzification [25]–[28]. A novel HA-based forecasting model was proposed in [29], which maps linguistic terms into fuzzy intervals to effectively determine interval lengths.

Handling uncertainties in time series forecasting remains a significant challenge, requiring robust optimization techniques [30], [31]. A variety of optimization methods have been employed, among which Particle Swarm Optimization (PSO) is widely used for its efficiency in global search. However, PSO often suffers from premature convergence and the trapping of local optima. To mitigate these limitations, Simulated Annealing (SA) serves as a complementary technique, enhancing exploration and allowing the model to escape local optima. A crucial factor influencing forecasting accuracy is the determination of interval lengths within the Universe of Discourse (UD). Hedge Algebra (HA), with its rigorous mathematical structure and linguistic processing capabilities, provides an effective means to divide the UD into intervals of varying lengths. Furthermore, the challenge lies in identifying the optimal interval partitions. To address this issue, we integrate HA with hybrid optimization techniques, leveraging PSO for global search and SA for local refinement, to dynamically adjust interval lengths. This hybrid approach ensures that the interval partitions are not only consistent with linguistic characteristics but also optimized for improved forecasting performance.

This paper introduces a novel forecasting model that synergizes HA's linguistic-based interval partitioning with the optimization strengths of PSO and SA. By utilizing HA to establish an adaptive interval structure and employing PSO-SA to fine-tune interval lengths, our approach enhances the effectiveness of fuzzy time series forecasting. Experimental evaluations on three benchmark datasets—University of Alabama enrollments, traffic accident fatalities in Belgium, and RON95 gasoline prices in Vietnam demonstrate that our proposed model outperforms existing approaches, validating the effectiveness of this integrated methodology.

The rest of this paper is structured as follows: Section 2 briefly introduces fundamental theories related to time series forecasting models, including FTS, HA, PSO, and SA algorithms. Section 3 delves into the detailed development of the FTS model based on aggregated HA and two optimization algorithms. Section 4 evaluates the effectiveness of the proposed forecasting model and compares its results with existing models. Conclusions and future work are given in Section 5.

2. Method

In this section, the basic theories related to the paper are summarized, including fuzzy time series [2], hedge algebras [24], and optimization algorithms. Based on these theories, we propose a forecasting model to solve real-world problems.

2.1. Preliminaries

2.1.1. Basic definitions related to FTS

In light of works by Zadel, Song and Chissom [2] have proposed the definition of fuzzy time series, and several related concepts were further improved by Chen [10] and N.C. Dieu et al. [32] as follows:

Let U denote the universe of discourse, where $U = \{u_1, u_2, \dots, u_n\}$. A fuzzy set A_i of U can be defined as follows:

$$A_i = \frac{\mu_{A_i}(u_1)}{u_1} + \frac{\mu_{A_i}(u_2)}{u_2} + \cdots + \frac{\mu_{A_i}(u_n)}{u_n} \quad (1)$$

where $\mu_{A_i}: U \to [0,1]$ is the membership function of A_i and $\mu_{A_i}(u_i)$ indicates the degree of membership of u_i in the fuzzy set A_i for $1 \le i \le n$. The symbol "+" denotes the operation of union.

2.1.1.1. Fuzzy time series [2]

Let the universe of discourse, $Y(t)$, be a subset of real numbers $\Re$, on which the fuzzy sets $f_i(t)$ (for i $= 1,2\ldots$) are defined, where $t = 0,1,2,\ldots,$. If $F(t)$ is a collection of $f_1(t), f_2(t), \ldots f_i(t),\ldots$, then $F(t)$ is called a fuzzy time series defined on $Y(t)$.

2.1.1.2. Fuzzy logical relationship (FLR) [2]

If there exists a fuzzy logical relationship $R(t-1,t)$ such that $F(t)$ may be expressed as $F(t) = F(t-1) * R(t-1,t)$, where "$*$ " denotes the max–min operator, then $F(t)$ is considered to be influenced by $F(t-1)$. This causal relationship between $F(t)$ and $F(t-1)$ may be represented as $F(t-1) \rightarrow F(t)$. Let $A_i = F(t)$ and $A_j = F(t-1)$; thus, the relationship between $F(t)$ and $F(t-1)$ may be described by fuzzy logical relationship $A_i \rightarrow A_j$ where A_i and A_j refer to the current state or the left-hand side and the next state or the right-hand side of fuzzy logical relationship.

2.1.1.3. p- order fuzzy logical relationship [15]

Let $F(t)$ be a fuzzy time series. If $F(t)$ is influenced by $F(t-1), F(t-2), \ldots, F(t-p+1), F(t-p)$ then this fuzzy logical relationship is represented as $F(t-p), \ldots, F(t-2), F(t-1) \rightarrow F(t)$ and is called an p- order fuzzy time series. Here, p denotes the order of fuzzy logical relationship.

2.1.1.4. Time-Dependent Fuzzy Relationship Group (TD-FRG) [32]

If $F(t)$ is influenced by $F(t-1)$, the FLR between them is represented by $F(t-1) \rightarrow F(t)$. Let $F(t) = A_i(t)$ and $F(t-1) = A_j(t-1)$. The FLR between $F(t-1)$ and $F(t)$ can be expressed as $A_j(t-1) \rightarrow A_i(t)$. Additionally, at time t, there exist fuzzy logical relationships as follows: $A_j(t1-1) \rightarrow A_{i1}(t1), \ldots, A_j(t\lambda - 1) \rightarrow A_{i\lambda}(t\lambda)$ with $t1, t2, \ldots, t\lambda \leq t$. Here, $t1$, $t2$, ...represent forecasting points. It is important to note that $A_{i1}(t1)$, $A_{i2}(t2), \ldots, A_{i\lambda}(t\lambda)$, occur at distinct time $t1, t2,\ldots, t\lambda$, respectively. This implies that if these FLRs occur before $A_j(t-1) \rightarrow A_i(t)$, we can group them into a Fuzzy Relationship Group based on the left-hand side of each fuzzy logical relationship as $A_j(t-1) \rightarrow A_{i1}(t1), A_{i2}(t2), \ldots, A_{i\lambda}(t\lambda), A_i(t)$. This is termed as first-order TD-FRG.

2.1.2. A Brief Introduction of Hedge Algebras

Each linguistic domain of variable, denoted by Dom($\mathcal{X}$), consists of a word set that can be generated from two generator words, e.g., "low" and "high", by the action of linguistic hedges on them. For instance, with two hedges of "very" and "little", the words generated by the action of those hedges can be "very low", "little low", "very high", "very very high", and so on. It can be seen that they are arranged in a linear order and can be compared as follows:

$$\mathcal{X} = \{Very\ Very\ low < Very\ low < Low < Little\ low < Very\ Little\ low < Medium < Very\ Little\ high < Little\ big < High < \cdots\} \tag{2}$$

Observing the above meaning, Nguyen Cat Ho [24] proposes the concept of hedge algebras with a algebraic structure $\mathcal{AX} = (X, G, C, H, \leq)$ and called HA, where X is the set of terms in $\mathcal{X}$; $\leq$ denotes a natural semantically ordering relation on X; $G = \{c^-, c^+\}$, $c^- \leq c^+$, is the set of primary generators, in which c^- and c^+ are the negative primary term and the positive one of $\mathcal{X}$, respectively; $C = \{0, 1, w\}$ a set of constants, with $(0 \leq c^- \leq W \leq c^+ \leq 1)$; $H = H^- \cup H^+$ with $H^- = \{h_{-q} \geq \cdots \geq h_{-2} \geq h_{-1}\}$ is the set of all negative hedges of X, $\forall h \in H^-$ then $hc^+ \leq c^+$ and $H^+ = \{h_1 \leq h_2 \leq \cdots \leq h_p\}$ is the set of all positive ones of X, $\forall h \in H^+$ then $hc^+ \geq c^+$. Some general definitions of HA are given as follows:

Let $\mathcal{AX} = (X, G, C, H, \leq)$ be a HA. A function fm: $X \rightarrow [0,1]$ is considered a fuzziness measure of terms in X if:

- $fm(c^-) + fm(c^+) = 1$ and $\sum_{h \in H} fm(hx) = fm(x)$ for $\forall x \in X$

- For the constants 0, W and 1, $fm(0) = fm(W) = fm(1) = 0$
- For $\forall x, y \in X, \forall h \in H$, $\frac{fm(hx)}{fm(x)} = \frac{fm(hy)}{fm(y)}$, that is this proportion does not depend on specific elements and it is called fuzziness measure of the hedge h and denoted by $\mu(h)$. The properties of $fm(x)$ and $\mu(h)$ are further elucidated as follows.

2.1.2.1. Proposition

Let fm is the fuzziness measure function on X, the following statements hold. With $x \in X$, for $x = h_n h_{n-1} \dots h_1 c$, where $h_j \in H$ and $c \in G$.

- $fm(hx) = \mu(h) fm(x), \forall x \in X$
- $\sum_{-q<i<p, i\neq 0} fm(h_i c) = fm(c)$
- $\sum_{-q<i<p, i\neq 0} fm(h_i x) = fm(x)$
- $fm(x) = fm(h_n h_{n-1} \dots h_1 c) = \mu(h_n)\mu(h_{n-1}) \dots \mu(h_1) fm(c)$
- $\sum_{i=-1}^{-q} \mu(h_i) = \alpha$ and $\sum_{i=1}^{p} \mu(h_i) = \beta$, where $\alpha, \beta > 0$ and $\alpha + \beta = 1$

The structure of the proposed model comprises two primary stages, illustrated in Fig.1

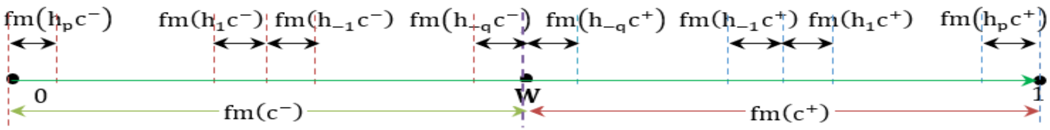


Fig. 1. The order of elements $x \in X, h_j \in H, c \in G$

2.1.3. Particle swarm optimization

PSO (Particle Swarm Optimization) was proposed by Kennedy and Eberhart [33] as a probabilistic search technique, inspired by the behavior and interactions of bird flocks or shoals of fish in their search for food. A population consisting of P individuals is represented by each particle, with each particle being characterized by two components: position X_{id} and velocity V_{id}. Each individual represents a potential solution, which is evaluated through an objective function (fitness function). Initially, random position and velocity vectors are assigned to initialize PSO. In each subsequent iteration of the algorithm, the velocity vector V_{id} and position X_{id} of each individual are updated according to equations (3) and (4). Furthermore, in every iteration, each individual is influenced by two pieces of information: the first is the best position it has achieved up to the current time, referred to as P_{best_id}, and the second is the best position achieved by any individual in the entire population during the search process up to the current time, referred to as G_{best}.

$$V_{id}^{t+1} = \omega^t * V_{id}^t + c_1 * Rand1() * (P_{best\_id} - X_{id}^t) + c_2 * Rand2() * (G_{best} - X_{id}^t) \tag{3}$$

$$X_{id}^{t+1} = X_{id}^t + V_{id}^{t+1} \tag{4}$$

Where:

X_{id}^t, V_{id}^t : are the current position and velocity of individual *id* at iteration *t*

c_1 and c_2 : are two learning factors which control the influence of the social and cognitive components (typically c_1 = c_2= 2)

$Rand1$ và $Rand2$: are two uniformly distributed random numbers in the range [0, 1]

ω^t : is the time-varying inertia weight. The parameter ω^t is calculated by (5):

$$\omega^t = \omega_{max} - \frac{t*(\omega_{max} - \omega_{min})}{t_{max}} \tag{5}$$

where ω_{max}, ω_{min} are the maximum and minimum values of the inertia weight, and t_{max} is the maximum iteration number.

2.1.4. Simulated annealing(SA)

Simulated Annealing [34] is an algorithm inspired by the annealing process in metallurgy. SA starts at a high temperature (T_0) when the metal is in a molten state. As the temperature gradually decreases to the ambient temperature after removing the heat source, the energy of the metal reaches its minimum, and the metal solidifies. The pseudocode of the SA algorithm (Algorithm 1)aimed at minimizing energy is briefly summarized as follows:

The standard SA algorithm

Initialize an energy state E_k with the initial temperature T_0 and cooling rate α ∈ [0, 1]. T = T_0,

Set T = T_0, where T_0 is the initial temperature.

Repeat until termination condition is met

  Compute the energy change ΔE between the current state E_i and the previous state E_k

  ΔE = E_i - E_k

  if ΔE < 0 then

  Accept the new state (E_k = E_i) (uphill move)

  else

  Accept the new state with probability $P = e^{-\frac{\Delta E}{C_b T}}$, where C_b is the Boltzman constant

  Generate a random number r in [0, 1]

  if r < P then

  Accept the new state (E_k = E_i)

  Reduce the temperature: T = α * T

End

2.2. A Proposed FTS Forecasting Model Combining HA with Optimization Techniques

This section introduces a hybrid prediction model utilizing FTS, combining optimization techniques and hedge algebras to enhance forecast accuracy. The proposed model's structure comprises two primary stages, as illustrated in Fig. 1. In the first stage, hedge algebras are utilized to partition the historical data into intervals and establish the FTS model, as detailed in Subsection 2.2.1. The second stage employs two optimization techniques, particle swarm optimization and simulated annealing, to determine the optimal interval lengths, as detailed in Subsection 2.2.2. The structure of the proposed forecasting model is outlined in Fig. 2 .

2.2.1. A proposed forecasting model based on FTS and HA

This section introduces a forecasting model based on FTS and hedge algebra to forecast the University of Alabama enrollments [2]. In this model, the HA concepts are employed to partition the universe of discourse (UD) of time series into initial intervals of different lengths by quantitatively mapping linguistic terms into fuzzy intervals. The forecasting model is established according to the following steps:

Step 1: Define the universe of discourse U of historical data

Let $U = [D_{min} - D_1, D_{max} + D_2]$ represents the universe of discourse. To define U, we use the minimum value $\mathrm{D_{min}}$ and the maximum value $\mathrm{D_{max}}$ of the historical time series data. Two positive integers, D_1 and D_2 are selected to ensure that the forecasting values are bounded within U. For historical enrollment time series, U is defined as U = [13000, 20000], where D_{min} = 13055, D_{max} = 19337, D_1 = 55, $\mathrm{D_2}$ = 663, $LU = (D_{max} + D_2) - (D_{min} - D_1)$ = 7000, is the length of U.

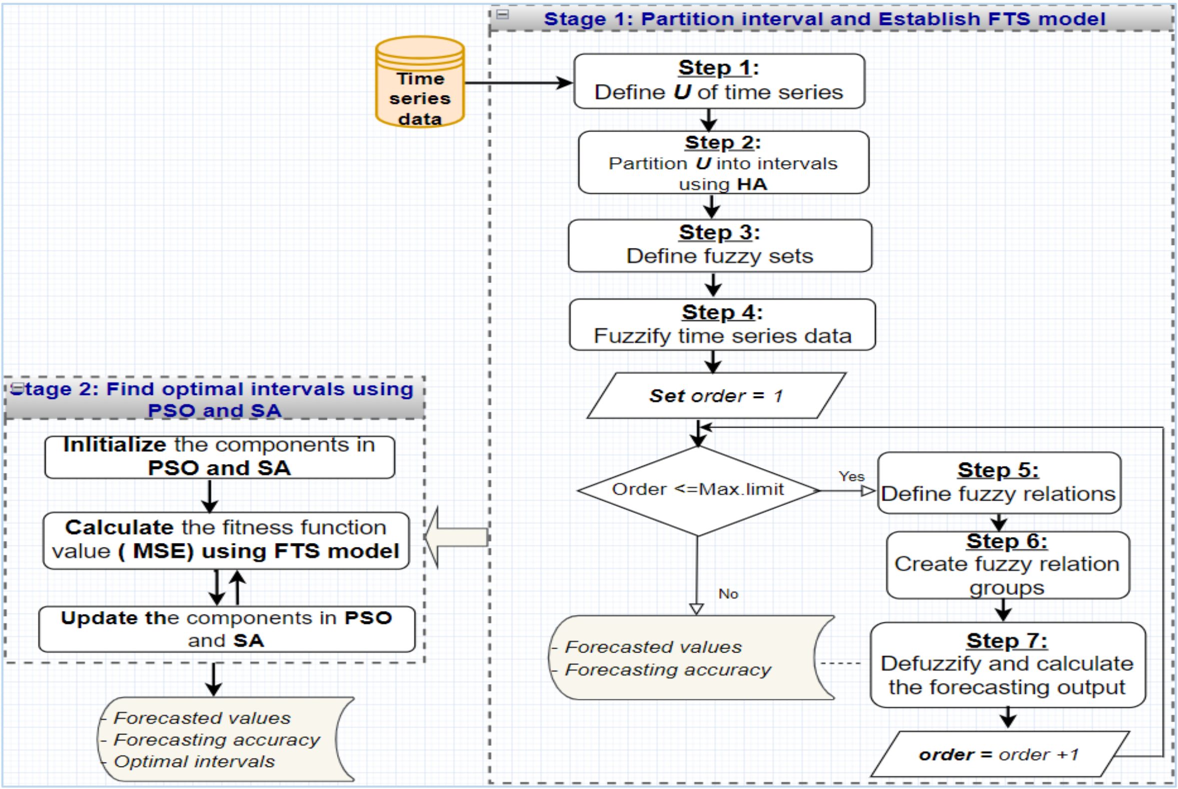


Fig. 2. Illustrates the structure of the proposed forecasting model

Step 2: Divide *U* into different intervals based on HA

This step employs HA with structure as $\mathcal{A}X = (X, G, C, H, \leq)$, where X is the set of terms of the linguistic variable "*enrollments*"{ X= dom(*enrollments*)}; ≤ denotes a natural semantic ordering relation on X; $G = \{c^-, c^+\}$ ={*Few, Many}, Few (Fw)* ≤ *Many* (*Ma*); $C = \{0, w, 1\}$ a set of constants, where $(0 \leq c^- \leq W \leq c^+ \leq 1)$ and H ={ *Very, Little* }. To facilitate the comparison of the forecasted results of the proposed model with other models, this study uses 7 intervals, corresponding to the number of linguistic terms used to quantify the time series values. Specifically, based on the linguistic terms given in Table 1, the corresponding intervals in the domain *U* are defined as shown in Substeps 2.1 and 2.2:

Table 1. The number of language terms

Number of linguistic terms	Terms of language and their sequence
7	A_1 = *Very Very Few (VVFw)* < A_2= *Little Verry Few (LVFw)* < A_3= *Little Little Few (LLFw)* < A_4=*Very Little Few (VLFw)* < A_5 =*Very Little Many (VLMa)* < A_6= *Little Little Many (LLMa)* < A_7= *Very Many (VMa)*

Step 2.1: The domain U = [13000, 20000] is mapped to the domain [0, 1]

Assume that the value of 16807 in the time series dataset corresponds to the Few value. The fuzziness measure of terms is calculated as follows: $fm(Few) = \frac{16807-13000}{20000-13000} = 0.544$, $fm(Many)$ = 1- 0.54 = 0.456 and LU = 20000 -13000 = 7000.

Mapping these values to U, we determine: covfm(*Few*) = $fm(Few)$ x LU = 0.544 x 7000 = 3808, and covfm(*Many*) = $fm(Many)$ x LU = 0.456 x 7000 = 31920. In this study, $\mu(Little)$ = 0.48, $\mu(Very)$ = 1- 0.48 = 0.52. From the values of $\mu(Little)$, $\mu(Very)$, the value of α, β are chosen as 0.48, 0.52, respectively. From here, the fuzziness interval of linguistic terms in the domain [0,1] are calculated as: $fm(VVFw)$ = 0.1471, $fm(LVFw)$ = 0.1358, $fm(LLFw)$ = 0.1253, $fm(VLFw)$ = 0.1358, $fm(VLMa)$ = 0.11138, $fm(LLMa)$ = 0.1051, $fm(VMa)$ = 0.2371.

Step 2.2: Define the fuzzy interval of the linguistic variable in the universe of discourse

Based on Step 2.1, the linguistic values of terms belonging to the fuzziness interval are calculated as follows:

$covfm(A_1) = \mu(Verry)\ x\ \mu(Very)\ x\ covfm\ (Fw) = 0.52*0.52*3808 = 1029.683;$
$covfm(A_2) = \mu(Little)\ x\ \mu(Very)\ x\ covfm\ (Fw) = 0.48*0.52*3808 = 950.477;$
$covfm(A_3) = \mu(Little)\ x\ \mu(Little)\ x\ covfm(Fw) = 0.48*0.48*3808 = 479.36;$
---;

$covfm(A_7)$ = $\mu(Very)$ x $covfm(Ma)$ = 0.52*3192 = 1659.84

The intervals corresponding to the linguistic terms are obtained by mapping the values of linguistic terms to the domain U as follows:

For seven linguistic terms, the seven intervals are: u_1= [13000, 14029.68), u_2= [14029.68, 14980.2), u_3= [14980.2, 15857.5), u_4= [15857.5, 16808), u_5= [16808, 17605), u_6= [17605, 18340.16), u_7= [18340.16, 20000].

Step 3: Define linguistic terms A_i represented by fuzzy sets

With seven intervals in Step 2, there are seven linguistic values representing distinct partitioning within the UD on U. Each linguistic value corresponds to a fuzzy set A_i, defined by the following formulas:

$$\mathrm{A_i} = {}^{a_{i1}}/\mathrm{u_1} + {}^{a_{i2}}/\mathrm{u_2} + \ldots + {}^{a_{ij}}/\mathrm{u_j} + \cdots + {}^{a_{i7}}/\mathrm{u_7} \tag{6}$$

$$a_{ij} = \begin{cases} 1 & j = i \\ 0.5 & j = i-1 \ or\ j = i+1 \\ 0 & otherwise \end{cases} \tag{7}$$

Here, the symbol '+' represents the set union operator, $a_{ij} \in [0,1]$ (for $1 \le i \le 7$, $1 \le j \le 7$), u_j denotes the j[th] interval of the UD. The value of a_{ij} denotes the degree of membership of u_j in the fuzzy set A_i. For simplicity, the membership values of the fuzzy set A_i are chosen based on a triangular membership function, as shown in (6). The corresponding linguistic values based on the obtained intervals from the enrollment dataset are illustrated in Fig.3.

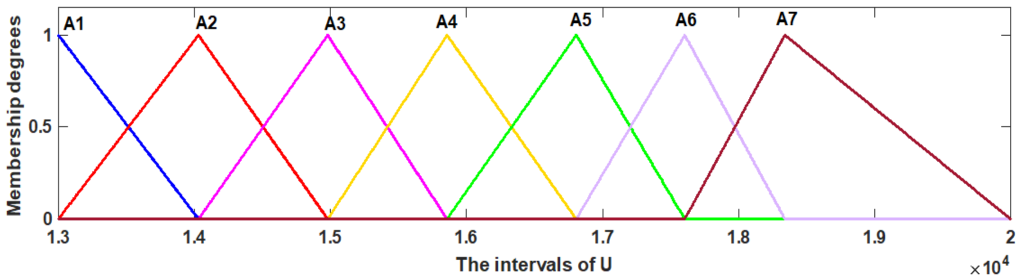


Fig. 3. The fuzzy sets are defined by intervals using the triangular membership function

Step 4: Fuzzy the historical time series data

To fuzzify the historical data, it is essential to obtain the degree of membership for each data value within each interval u_i for each year. If the maximum membership value of a day's observation occurs at u_i (where $1 \le i \le 7$), then the fuzzified value for that particular year is considered to be A_i. For example, if the historical enrollment of the year 1972 is 13563, and it belongs to interval u_1 because 13563 falls within [13000, 14029.68), we assign the linguistic value "$Very\ Very\ Few$" (eg. the fuzzy set A_1) corresponding to interval u_1 to it. According to (6), the fuzzy set A_1 has the maximum membership value at the interval u_1. Therefore, the historical enrollment data for the year Y(1972) is fuzzified to A_1. The completed fuzzified results of the enrollment data are presented in Table 2.

Table 2. Fuzzified historical enrollments data of the University of Alabama

Year	Real data	Fuzzy sets	Linguistic values
1971	13055	A_1	*VVFw*
1972	13563	A_1	*VVFw*
---	---	---	-------
1991	19337	A_7	*LLMa*
1992	18876	A_7	*LLMa*

Step 5: Create the p– order fuzzy logical relationships (p → 1)

Based on the concept of p-order fuzzy logical relationships above, to establish a p-order fuzzy logical relationship, we need to identify any relationship in the form of $F(t-p), F(t-p+1), \ldots, F(t-1) \to F(t)$, here $F(t-p), F(t-p+1), \ldots, F(t-1)$ are referred to as the current state, and $F(t)$ is referred to as the next state. A p-order fuzzy relationship in the training phase is obtained by replacing the corresponding linguistic values.

For instance, suppose $p = 1$, from Table 2, a fuzzy relation $A_1 \to A_1$ is derived as $F(1971) \to F(1972)$. Consequently, the first-order FLRs are shown in column 4 of Table 3, where there are 22 fuzzy logical relationships. The first 21 relationships are termed trained patterns, and the last one is termed an untrained pattern (in the testing phase). For the untrained pattern, relationship 22 has the fuzzy relation $A_7 \to \#$; it is created by the relationship $F(1992) \to F(1993)$, with the linguistic value of $F(1993)$ is unknown within the historical data, and this unknown next state is denoted by the symbol '#'. Similarly, suppose $p = 2$. A fuzzy logical relationship $(A_1, A_1) \to A_1$ is obtained as $(F(1971), F(1972)) \to F(1973)$. The complete second-order fuzzy relationships are listed in column 5 of Table 3.

Table 3. The complete the first-order and the second - order FLRs

No	Year	Fuzzy set	1st-order FLRs	2rd- order FLRs
1	1971	A_1		
2	1972	A_1	$A_1 \to A_1$	
3	1973	A_1	$A_1 \to A_1$	$A_1, A_1 \to$ A1
4	1974	A_2	$A_1 \to A_2$	$A_1, A_1 \to A_2$
---	---	---	---	---
21	1992	A_7	$A_7 \to A_7$	$A_7, A_7 \to A_7$
22	1993	#	$A_7 \to \#$	$A_7, A_7 \to \#$

Step 6: Establish all time-dependent fuzzy relationship groups (TD-FRGs)

In this step, we rely on the concept of time-dependent fuzzy relationship groups [32], as presented in Definition 4, to create FRGs. To explain this, we consider three first - order FLRs at three different times t = 1972, 1973, and 1974 which are presented in column 4 of Table 3 as follows: at time t = 1972, there is an FLR $A_1 \to A_1$; at time t =1973, there is an FLR $A_1 \to A_1$; at time t =1974, there is an FLR $A_1 \to A_2$. All of these have the same fuzzy set A_1 on the left – hand side.

Considering the forecasting time t = 1992, we obtain a first - order FRG (*i.e., Group 1*) as follows: $A_1 \to A_1$. For the forecasting time t = 1993, there are two FLRs with the same the left - hand side, which can be grouped into an FRG as *Group 2*: $A_1 \to A_1, A_1$. For the forecasting time t = 1994, *Group 3* is expressed as follows $A_1 \to A_1, A_1, A_2$. A similar explanation applies to high-order fuzzy logical relationships ($p \geq 2$). From this explanation, we complete all first – order and second – order TV-FRGs as shown in column 2 and column 3 of Table 4, respectively.

Table 4. The complete the first – order and second –order TD-FRGs

Group	1st - order TD-FRGs	2rd - order TD-FRGs
1	$A_1 \rightarrow A_1$	$(A_1,\ A_1) \rightarrow A_1$
2	$A_1 \rightarrow A_1,\ A_1$	$(A_1,\ A_1) \rightarrow A_1,\ A_2$
--	---	---
20	$A_7 \rightarrow A_7,\ A_7$	$(A_7,\ A_7) \rightarrow A_7,\ A_7$
21	$A_7 \rightarrow A_7,\ A_7, A_7$	$(A_7,\ A_7)) \rightarrow \#$
22	$A_7 \rightarrow \#$	

Step 7: Defuzzification and calculate the forecasting output values

The final step of the proposed model involves defuzzifying the forecasting state to a crisp output value based on fuzzy forecasting rules. In this step, we introduce a defuzzification principle to compute the forecasted value for all first–order and high–order TD-FRGs during the training phase, as shown in Equation (8). Next, we use a defuzzification principle [35] to handle the unknown linguistic value during the testing phase, as shown in (9). The forecasting principles are outlined as follows.

Principle 1: Calculation of forecasting output value in the training phase

Assume the fuzzified data of year t -1 is A_j , and there is a first–order fuzzy relationship group where the current state and next states are A_j and A_{i1}, A_{i2}, A_{in} respectively, as follows: $A_j(t-1) \rightarrow A_{i1}(t1), A_{i2}(t2), A_{in}(tk)$.

The forecasting output value for time t, based on information on the right–hand side in each group, is defined as follows:

$$Forecasted_{Value} = \frac{index\_A_{i1} \times m_{i1} + index\_A_{i2} \times m_{i2} + \cdots + iindex\_A_{in} \times m_{in}}{\sum_{k=1}^{n} index\_Aik} \tag{8}$$

Where m_{i1}, m_{i2}, m_{in} are the middle values of the intervals u_{i1}, u_{i2} and u_{in} respectively, $index\_A_{i1}, index\_A_{i2}, \ldots, index\_A_{in}$ are the indexes of the k^{th} fuzzy set in the TD-FRGs, and n denotes the total number of fuzzy sets on the next state of TD-FRGs.

Principle 2: Calculation of forecasting output value in the testing phase

Assume there is a p - order fuzzy relationship group as $A_{t-p}, A_{t-(p+1)}, A_{t1} \rightarrow \#$. The forecasting value in the testing phase is estimated according to (9), where the symbol w_h represents the highest votes predefined by the user for each problem, p is the order of the FLRs, the symbols M_{t-1}, M_{t-2}... and $M_{t-\mathrm{p}}$ are the middle values of the corresponding intervals related to the latest fuzzy set and other fuzzy sets on the left-hand side of the fuzzy relationship group, which have the maximum membership values of A_{t-1}, A_{t-2}, ..., and A_{t-p} occurring at intervals u_{t1}, u_{t2},..., and u_{t-p}, respectively.

$$Forecated\_\text{value} = \frac{(M_{t-1} * w_h) + M_{t-2} + \cdots + M_{t-\mathrm{p}}}{w_h + (\mathrm{p}-1)} \tag{9}$$

Based on the two forecasting principles above and the fuzzy relationship groups in Table 4, we completed the forecasting results for enrollments from 1971 to 1992 based on first-order and high–order TD-FRGs under seven intervals, as shown in columns 4 and 5 of Table 5, respectively.

The efficiency of the proposed model is evaluated using various statistical indexes, namely Mean Square Error (MSE) and Mean Absolute Percentage Error (RMSE). The evaluation criteria are determined by the following equations.

$$MSE = \frac{1}{n} \sum_{i=p}^{n} (x_i - f_i)^2 \tag{10}$$

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=p}^{n}(x_i - f_i)^2} \quad (11)$$

where, x_i denotes the the actual time series and f_i is the forecasted time series; n is the total number of years to be forecasted, p is the order of fuzzy logical relationship.

Table 5. The complete forecasted output values based on the first–order and second-order FLRs

Year	Actual data	Fuzzy set	1st – order	2rd – order
1971	13055	A_1	---	---
1972	13563	A_1	13600.6	---
1973	13867	A_1	13772.3	13943.9
1974	14696	A_2	14095.7	14343.2
---	---	---	---	---
1991	19337	A_7	19308.4	19308.4
1992	18876	A_7	19124	19031.8
MSE			**129623.34**	**70188.37**

2.2.2. A hybrid FTS forecasting model combining HA and PSO-SA

In this section, a hybrid FTS forecasting model combining HA and optimization techniques is presented to improve forecasting accuracy. The model integrates PSO and SA algorithms to minimize forecasting error (MSE) by dynamically adjusting the lengths of partitioning intervals determined by HA. This optimization enhances time series representation and reduces errors, ensuring better forecasting performance. By leveraging PSO and SA, the model efficiently selects optimal interval boundaries, mitigating the impact of poor initial partitions.

The proposed forecasting model is named FTSHA-PSOSA and is briefly explained as follows. In the FTSHA-PSOSA model, during the training phase, each particle represents the partitioning of time series data (e.g., n intervals). Assume that the lower bound and upper bound of U are x_0 and x_n , respectively. Each particle in PSOSA is denoted as a vector containing $n-1$ elements: x_1, x_2, ..., x_{n-2} and x_{n-1}, where $(1 \leq id \leq n-1)$ and $x_{id} \leq x_{id+1}$. From these *n*-1 elements, define the *n* intervals as $u_1 = [x_0, x_1]$, $u_2 = [x_1, x_2]$,,..., $u_{id} = [x_{id-1}, x_{id}]$,... and $u_n = [x_{n-1}, x_n]$, respectively. In the case of a particle *id* moving from one position to another in the swarm, the elements of the corresponding new array must always be adjusted in ascending order such that $x_1 \leq x_2 \leq \cdots \leq x_{n-1}$, while the new velocity and position values are updated according to equations (1) and (2) until this condition is satisfied. If the stopping requirement is met, then all of the TD-FRGs acquired by the global best position (Gbest) among all personal best positions (Pbest) of all particles are used to predict the new testing data during the testing phase. The following steps outline the proposed model, as shown in Algorithm 1.

Algorithm 1: Structure of the proposed FTS forecasting model by combining HA with two optimization techniques PSOSA, simultaneously

1. **Input**: Historical time series data
2. **Output**: The forecasting results and the MSE value (MSE = *Gbest* = min(*Pbest*))

Begin

3. Define the initial intervals by applying HA and **use** forecasting steps in Subsection 3.1 to achieve the initial forecasting accuracy (MSE).

4. **Initialize:**

A random swarm with P particles and all necessary variables such as: cycle step t, initial temperature T0, and cooling rate α.

The initial position X_{id} and the velocity V_{id} of all particles, respectively.

The initial personal best position vectors of the particle *id* are the same as their initial position vectors, and find *Gbest*

5: **Repeat**

5.1: **for each** particle *id* in swam, ($1 \leq id \leq P$), **do:**

Follow the steps in Subsection 3.1 sequentially, from step 3 to step 7, including: defining linguistic terms, fuzzifying all historical data, determining all p − order fuzzy logical relationships, establishing all p − order TD-FRGs, defuzzifying forecasting values.

Estimate the objective function values for particle X_{id} based on function MSE(10)

5.2: **for each** all particle *id*, ($1 \leq id \leq P$), **do:**

Update the velocity according to Eq.(1)

Update the position according to Eq.(2)

Compare the objective values at the new position X_{id}^{t+1} and the old position X_{id}^{t}. If X_{id}^{t+1} is better, accept it as the new position of X_{id}; otherwise, compute the excess ΔE between X_{id}^{t+1} and X_{id}^{t} using the equation (12).

$$\Delta E = MSE_{id}^{t+1} - MSE_{id}^{t} \quad (12)$$

Generate a random number $\alpha \in [0, 1]$. Accept new position if $> e^{-(\frac{\Delta E}{T^t})}$.

If the new position is accepted, update the new P_{best_id} for each particle *id* and the global best *Gbest* based on the MSE values. Otherwise, go to Step 5.2

end for

Update ω according to equation (3)

Modify the annealing temperature $T^{t+1} = \alpha * T^t$, $t = t + 1$, and return to Step 5.1

until (t > the maximum iteration number (t_{max}) or the minimum MSE is reached)

return The best solution *(Gbest value)*

End.

Algorithm 2. The FTSHA-PSOSA algorithm in the testing phase

Use the optimal lengths of intervals and order of FLRs obtained in Algorithm 1 to estimate untrained data in the testing phase based on the Principle 2 in Subsection 3.1

3. Results and Discussion

In this section, three datasets are used to evaluate the effectiveness of the proposed model: (1) the University of Alabama enrollment data (1971–1992) [2], widely used in previous studies [2], [10], [12], [17], [19], [22], [32], [35]; (2) road traffic accident fatality data in Belgium [36]; and (3) the RON95 gasoline price dataset in Vietnam, collected from VNExpress (https://vnexpress.net/gia-xang-ve-duoi-22-000-dong-mot-lit-4779210.html). These datasets demonstrate the model's applicability to one-step-ahead forecasting, with results compared against existing forecasting models.

The forecasting model was implemented in C# and executed on an Intel Core i7 PC with 8GB RAM. Each dataset was processed through 20 independent runs, varying the order of fuzzy relationships and the number of intervals. The PSOSA parameters, as detailed in Table 6, were used to optimize forecasting performance. The best result from all runs is reported and benchmarked against previously published models. The performance of the proposed model is assessed using MSE and RMSE, as defined in Equations (10) and (11).

Table 6. The parameters of PSO and SA used in the FTSHA-PSOSA model

The parameters of PSO and SA	Enrollments	Car road accident	RON95
Number of particles *N* =	30	30	30
Max number of iterations *Tmax* =	150	150	150
Value of inertial weight ω	ω_{max}= 1.4 to ω_{min} = 0.4	ω_{max}= 1.4 to ω_{min} = 0.4	ω_{max}= 1.4 to ω_{min} = 0.4
Coefficient *C1*= *C2* =	2	2	2
The limited range of *V* =	[-100, 100]	[-50, 50]	[-100, 100]
The limited range of *X* =	[13000, 20000]	[953, 1644]	[21800, 25400]
Initial temperature T0	100	100	100
Cooling rate α	0.98	0.98	0.98

3.1. Experimental results for forecasting enrollments

3.1.1. Forecasting results based on the first – order FTS

To evaluate the performance of the suggested model based on first-order FTS with the number of intervals set to 7, comparisons were made with forecasting models published in studies [29], [37]–[39] . The RMSE values presented in Table 7 highlight its superior forecast-ing accuracy compared to existing models. Specifically, our model achieves the lowest RMSE of 191.9, demonstrating its ability to minimize forecasting errors. The distinction between the suggested model and the aforementioned models lies in the approach to constructing the fuzzy relationship group and the method of dividing the UD to establish the forecasting mod-el. Compared to previous studies, models in [36], [37] use Chen's framework with information granules, whereas our method leverages HA's structured linguistic processing to establish variable-length intervals. Additionally, the two models in papers [29], [39] are based on HA and Chen's fuzzy relationship groups for forecasting model structure but lack the optimiza-tion-driven refinement provided by our HA-PSO-SA approach. Furthemore, our model adopts an approach that leverages the concept of time-dependent fuzzy relationship groups [32] for model development. Finally, unlike the intuitionistic FTS model in [38] that employs fuzzy clustering for partitioning, our model combines HA's linguistic reasoning with PSO and SA to optimize intervals dynamically, resulting in superior forecasting precision. By effec-tively integrating HA with PSO for global search and SA for local tuning, our model signifi-cantly reduces forecasting errors, outperforming existing fuzzy time series approaches. The results confirm that this hybrid methodology enhances accuracy and provides a robust solu-tion for time series forecasting.

Table 7. A comparison of the forecasting outcomes of the FTSHA-PSOSA model with other models based on first-order FTS across seven intervals

Year	Real data	[37]	[40]	[36]	[39]	[38]	[29]	FTSHA-PSOSA
1972	13563	13486	13944	14279	13820	13500	13865	13464.5
---	---	---	---	---	---	---	---	---
1992	18876	18808	18933	19257	19135	18855	15219	19061.3
MSE		334430.9	256036	198203	194745.7	123130.8	44478.8	36825.6
RMSE		578.3	506	445.2	441.3	350.9	210.9	191.9

Additionally, the FTSHA-PSOSA model is executed 20 times and compared with several first-order FTS models referenced in studies [12], [19], [25], [32], [35], [36], [41], [42]. Table 8 shows that, with 14 intervals for all models, the FTSHA-PSO model achieves the lowest RMSE value of 74.2 among the evaluated models.

Table 8. An evaluation of the forecasting outcomes from the FTSHA-PSOSA model in comparison with other first-order FLR using 14 intervals

Year	real data	[12]	[19]	[42]	[35]	[36]	[41]	[32]	[25]	FTSHA-PSOSA
1971	13055									
1972	13563	13653	13714	13469	13555	13512	13579	13434	13598	13556.64
----	----	----	----	----	----	----	----	----	----	----
1992	18876	19059	19014	19152	19014	18718	19031	18820	18981	18957
MSE		31684	35324	23709	22965	14534	8224	7475	6332	**5500.5**
RMSE		178	187.9	153.98	151.5	120.6	90.7	86.5	7957	**74.2**

3.1.2. Forecasting results based on the high – order fuzzy time series

In this study, historical enrollment data from 1971 to 1992 [2] is split into two parts to compare the forecasting results of FTSHA-PSOSA model with other high-order models. The first part, consisting of 19 observations from 1971 to 1989, is used for training, while the second part, with three observations, is used for testing. The performance of our model and comparison models is evaluated using the MSE (10) and RMSE (11) metrics.

Case (1): Experimental results in the training phase

To highlight the forecasting accuracy of the FTSHA-PSOSA model with 7 intervals, results from studies [14], [19], [32], [35], [41] were compared, as shown in Table 9. The FTSHA-PSOSA model consistently achieved the lowest MSE values across different orders, ranging from 16358 for the 2nd-order to 1229.5 for the 9th-order. These results clearly indicate that the FTSHA-PSOSA model outperforms all other models across different high-order fuzzy relationships. Notably, the FTSHA-PSOSA model achieved the lowest MSE value of 846.4 under 4th-order fuzzy relationships. The key difference between the FTSHA-PSOSA model and the comparison models lies in how fuzzy relationship groups are established and the optimization method used. Specifically, model [19] employs a genetic algorithm, while models from [32], [35], [41] utilize the PSO algorithm to determine optimal intervals. Instead of using equal-length intervals, our proposed model incorporates hedge algebra to generate initial intervals with varying lengths and then combines both PSO and SA optimization techniques to continuously adjust interval lengths until the optimal intervals are found. The integration of PSO and SA in the FTSHA-PSOSA model offers a dual advantage: PSO facilitates efficient global search, while SA prevents the algorithm from being trapped in suboptimal solutions. Unlike single-method optimization approaches, this hybrid mechanism dynamically refines interval partitions, enhancing predictive accuracy.

Another distinction is that the FTSHA-PSOSA model forms fuzzy relationship groups based on time-dependent fuzzy relationships, whereas other models are structured according to Chen's framework [10]. According to the findings of the preceding investigation, the FTSHA-PSOSA model provides more compelling forecasting results than its counterparts based on the high-order FTS.

Table 9. An evaluation of the forecasting outcomes from the FTSHA-PSOSA model in comparison with other high-order FTS models utilizing 7 intervals

Order	[14]	[19]	[35]	[41]	[32]	FTSHA-PSOSA
2	89093	67834	67123	19594	19868	16358
3	86694	31123	31644	31189	31307	1896.4
4	89376	32009	23271	20155	23288	1246
5	94539	24984	23534	20366	23552	897.2
6	98215	26980	23671	22276	23684	846.4
7	104056	26969	20651	18482	20669	947
8	102179	22387	17106	14778	17116	1112
9	102789	18734	17971	15251	17987	1229.5
Average MSE	**95867.63**	**31377.5**	**28121.38**	**20261.38**	**22183**	**1878.79**

For clearer visualization, Fig. 4 illustrates the trend in forecasting accuracy between the FTSHA-PSO model and its counterparts across different high-order FLRs. The curves clearly show that, under all high-order FLRs, the proposed model consistently achieves the smallest forecasting error compared to the other models. Based on the provided examples, it can be concluded that the FTSHA-PSOSA model outperforms existing models in forecasting the University of Alabama enrollment numbers when utilizing high-order FTS with 7 intervals.

Case (2): Experimental results in the testing phase

Based on historical data from previous years, we may forecast the new enrollment for the upcoming year. For example, enrollment data from 1971 to 1989 is used to forecast enrollment for 1990, and similarly, data from 1971 to 1990 can be used to forecast enrollment for 1991. After the FTSHA-PSOSA model is thoroughly trained on the dataset, future enrollment values can be calculated and compared with those from forecasting models mentioned in references [10], [19], [35].

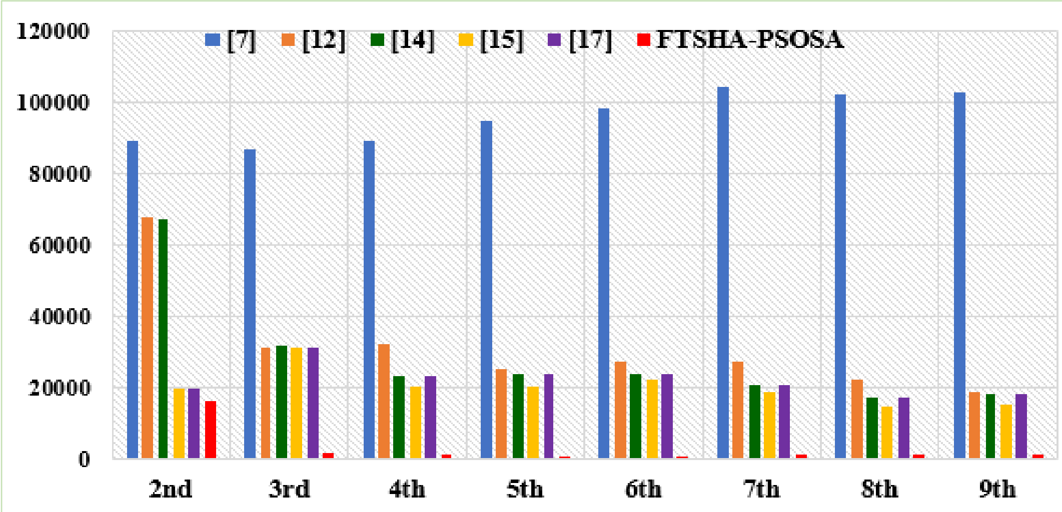
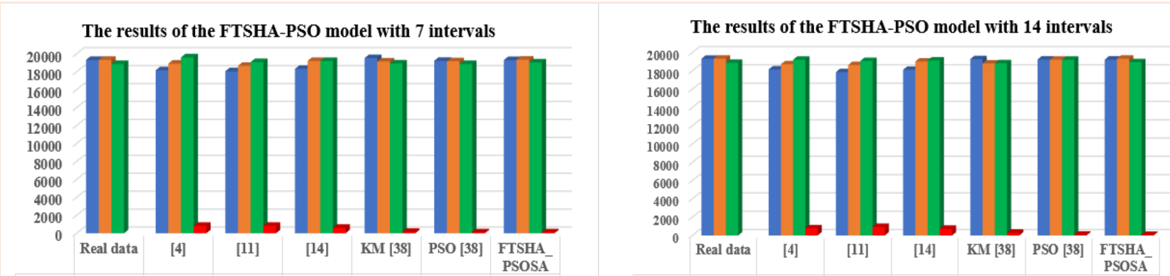


Fig. 4. The curves display the MSE values for the comparison between the FTSHA-PSOSA model and its counterparts based on different high-order FTS

The forecast results of the proposed model, which are based on the 3rd-order fuzzy logic rule (FLR) with different numbers of intervals and the highest vote W_h=20 are compared to other forecasting models and shown in Fig. 5. From these results, it can be observed that the FTSHA-PSOSA model achieves the smallest RMSE values of 98.6 and 72.53 based on the 7-interval and 14-interval, respectively, among the five compared models. These findings demonstrate that our model is more accurate than its counterparts, which are based on the 3rd-order fuzzy logic rule with varying numbers of intervals.



	Real data	[4]	[11]	[14]	KM [38]	PSO [38]	FTSHA_ PSOSA
1990	19328	18168	18059	18326	19525	19226	19308
1991	19337	18909	18669	19212	19150	19182	19351.9
1992	18876	19609	19083	19203	18933	18876	19045
RMSE		829.9	836.5	612.8	160.23	107.12	98.6

	Real data	[4]	[11]	[14]	KM [38]	PSO [38]	FTSHA_ PSOSA
1990	19328	18162	17862	18120	19287	19238	19230
1991	19337	18721	18633	19027	18811	19224	19336.74
1992	18876	19221	19085	19137	18836	19224	18954.6
RMSE		787	946.6	735.6	305.7	92.35	72.53

Fig. 5. A comparison of the forecasting results between the FTSHA-PSO model and other models with the number of intervals equal to 7 and 14, respectively, which use vote W_h=20

3.2. Experimental results for the "killed" in car road accidents data

In this section, the proposed model was employed to predict road traffic accidents in Belgium over the period from 1974 to 2004 [12]. Table 10 presents the forecasted values of the proposed model in comparison to those reported by research works [12], [25], [42]–[44]. As shown in Table 10, the proposed model outperforms other models. Notably, with 14 intervals, it achieves the lowest RMSE of 1.76, surpassing the third-order FTS model [43] and significantly outperforming the first-order FTS models [12], [25], [42]–[44], which used different numbers of intervals. It is essential to note that the primary distinctions between our model and the compared models stem from the optimization algorithm and interval partitioning method.

The forecast outcomes of the proposed model, using both the first-order and third-order fuzzy logical relationships, are depicted in Fig. 6. This figure highlights that the predicted values are closely aligned with the observed data, with the third-order model exhibiting higher predictive accuracy. Additionally, the results suggest a slight upward trend in car accident rates in Belgium projected for 2005. More importantly, the proposed model is not only effective in predicting traffic accidents but also highly flexible, making it applicable to various forecasting problems, such as enrollment data. Its adaptive optimization and efficient interval segmentation enable it to handle different datasets, expanding its potential applications across multiple domains. In summary, the proposed model not only outperforms existing methods in forecasting road traffic fatalities but also demonstrates strong adaptability, reinforcing its flexibility and effectiveness.

Table 10. Compare the output results of the proposed forecasting model with previous models based on first-order and third-order FLRs

Year	Real data	[12]	[44]	[42]	[25]	[43]	Proposed Model	
							1st_ order	*3rd_ order*
1974	1574							
1975	1460	1506	1458	1451	1464		1455.6	
1976	1536	1453	1467	1490	1515		1546.5	
1977	1597	1598	1606	1622	1610	1594	1586.2	1596.7
1978	1644	1584	1592	1575	1623	1643	1623.75	1643.8
-----	-----	-----	-----	-----	----	-----	-----	-----
2003	1035	970	1097	1028	1032	1036	1040.6	1036.9
2004	953	970	929	953	982	954	954.14	952.5
2005	N/A						955.8	954.3
RMSE		41.61	37.66	32.0	20.52	19.2	15.8	1.75

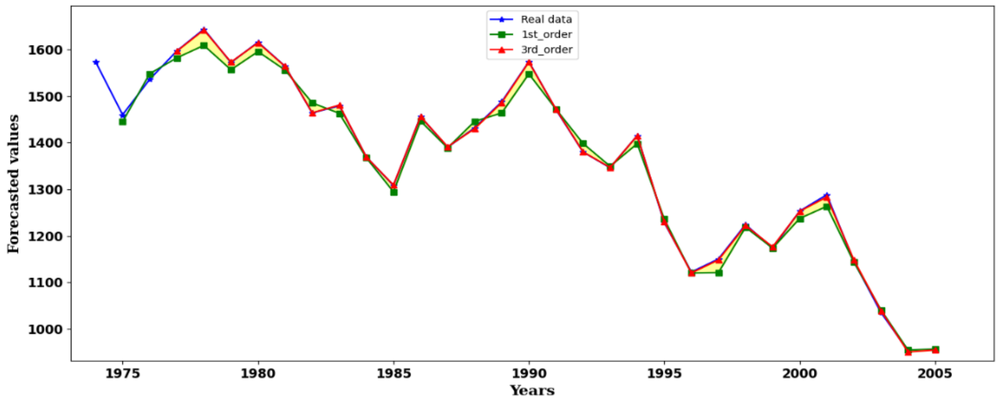


Fig. 6. The relationship between the real data and the forecast data based on the 1st-order and 3rd-order FLRs

3.3. Experimental results for the data of Gas price RON95 in Vietnam

Next, the proposed model is applied to forecast the price of RON95 gasoline in Vietnam from January 04, 2024, to August 02, 2024. The forecasting performance of the proposed model is evaluated based on the MSE criterion. The results and forecast errors of the proposed model, based on a first-order fuzzy relationship with 14 partitions, are shown in Table 11.

Table 11. The forecasting results of the FTSHA-PSOSA model based on the first–order FLRs with a number of intervals equal to 14

Date	Real RON95	FTSHA-PSOSA
04/01/2024	21910	
12/01/2024	21930	21872
----	----	----
01/08/2024	22600	22514.95
02/08/2024	N/A	22503
MSE		5980

Visually, the forecasting trends of the two forecasting models are illustrated in Fig. 7

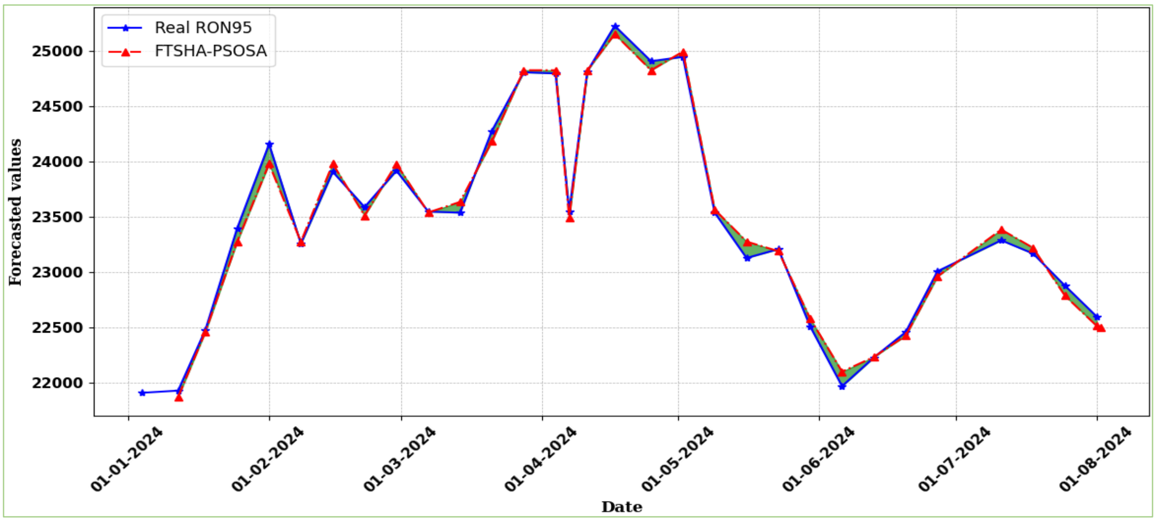


Fig. 7. The graph illustrates the relationship between the forecasted gasoline prices and the actual gasoline prices

The results presented in Table 11 and Fig. 7 clearly demonstrate that the forecasted values generated by the proposed model closely align with the actual RON95 gasoline prices. Such reliable forecasting performance is achieved due to the FTSHA-PSOSA model, which leverages the structure of Hedge Algebra to select partitions of varying lengths, combined with optimization methods to fine-tune and determine the optimal interval lengths.

To further demonstrate the superiority of the proposed high-order FTS model with 7 intervals, Table 12 presents the forecasting results in terms of MSE for different FLR orders, ranging from the 2nd to the 8th order. The 6th-order FLR stands out as the best, achieving the lowest MSE of 483.2 among all orders.

Table 12. Forecasting results of FTSHA-PSOSA using 7 intervals based on high-order FLR

Date	Real RON95	Forecasted values						
		2nd-order	3rd-order	4th-order	5th-order	6th-order	7th-order	8th-order
18/01/2024	22480	22472.3						
25/01/2024	23400	23393.2	23359.8					
01/02/2024	24160	24143.3	24211.5	24113				
08/02/2024	23260	23279.5	23193	23251.5	23292			
15/02/2024	23910	23915	23939.8	23913	23920.5	23917		
22/02/2024	23590	23543	23537.5	23562	23555	23553.7	23545.5	
29/02/2024	23920	23915	23939.8	23913	23920.5	23917	23934.5	23918
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01/08/2024	22600	22337	22584.7	22647.3	22649.5	22613.3	22599.7	22673.3
MSE		**13650.4**	**1158.8**	**717.7**	**515.8**	**483.2**	**569.6**	**625.5**

For ease of visualization, Fig. 8 shows the curves of actual and forecasted values for pre-dicting the price of RON95 gasoline in Vietnam from January 04, 2024, to August 02, 2024. It can be seen from this figure that the curve of the proposed model, based on the 7th-order FLR, is the closest to the actual data compared to all other orders.

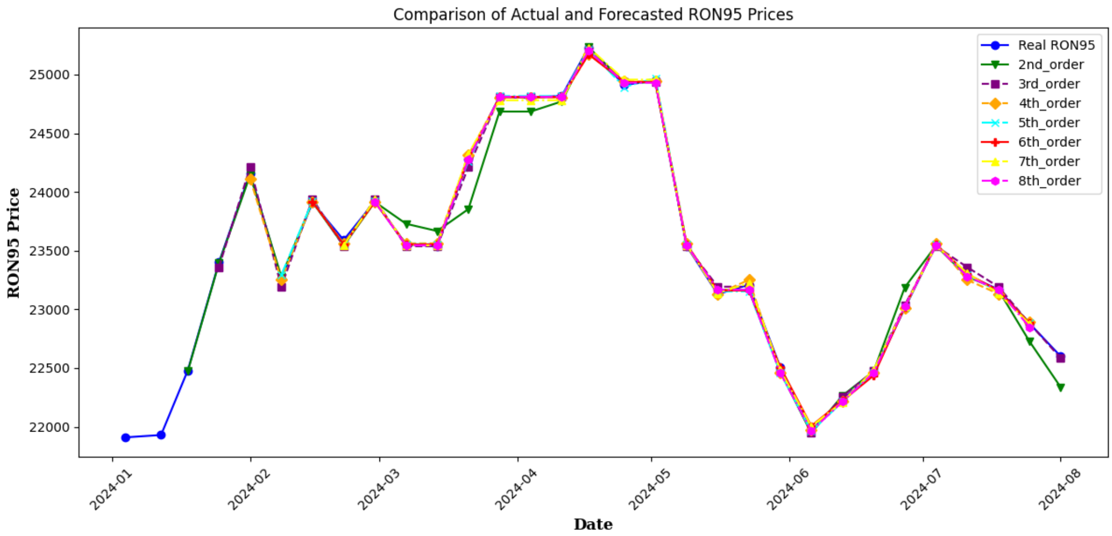

Fig. 8. The chart illustrates the trend between forecasted values and actual RON95 prices based on higher-order FLRs

4. Conclusion

In this study, a hybrid forecasting model, FTSHA-PSOSA, which integrates fuzzy time series with hedge algebra and optimization techniques, was developed. The FTSHA-PSOSA model effectively addresses two key factors that impact forecasting accuracy: interval length determination and the formation of fuzzy relationship groups. Traditional FTS models often encounter limitations when using time-invariant fuzzy relationship groups, whereas FTSHA-PSOSA overcomes these by utilizing time-varying fuzzy relationships to produce more accurate forecasts. The proposed model leverages the mathematical structure of HA to generate variable-length intervals and integrates PSO and SA to fine-tune the optimal interval length, further enhancing forecasting accuracy. Comparative analysis using datasets from the University of Alabama and Belgian car accidents demonstrates that FTSHA-PSOSA significantly outperforms existing models. Additionally, simulations on the RON95 gasoline price dataset in Vietnam further confirm its superiority, particularly in handling higher-order FLRs. However, despite these promising results, several limitations remain. The determination of FLRs in higher-order models is computationally intensive and requires further refinement. Additionally, the model's performance may vary depending on the dataset characteristics, necessitating adaptive mechanisms for parameter tuning. Future research should focus on automating the optimal order selection for higher-order FLRs and exploring advanced optimization techniques to further improve computational efficiency. Moreover, extending the model to diverse real-world applications, such as financial market predictions and climate forecasting, will be an important area of future study.

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Declarations

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