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An approach for linguistic multi-attribute decision making based on linguistic many-valued logic

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ABSTRACT

There are various types of multi-attribute decision-making (MADM) problems in our daily lives and decision-making problems under uncertain environments with vague and imprecise information involved. Therefore, linguistic multi-attribute decision-making problems are an important type studied extensively. Besides, it is easier for decision-makers to use linguistic terms to evaluate/choose among alternatives in real life. Based on the theoretical foundation of the Hedge algebra and linguistic many-valued logic, this study aims to address multi-attribute decision-making problems by linguistic valued qualitative aggregation and reasoning method. In this paper, we construct a finite monotonous Hedge algebra for modeling the linguistic information related to MADM problems and use linguistic many-valued logic for deducing the outcome of decision making. Our method computes directly on linguistic terms without numerical approximation. This method takes advantage of linguistic information processing and shows the benefit of Hedge algebra.

1. Introduction

Multi-attribute decision making (MADM) problems is selecting an optimal choice with the highest degree of satisfaction from a set of alternatives, which alternatives characterized on their attributes [1]. In practice, sometimes the information is vague and imprecise, which may lead to a decision made under uncertainty [2], [3]. In these situations, the information cannot be assessed precisely in a quantitative form but may be in a qualitative one, and thus, the use of a linguistic approach is necessary [4]. For example, when evaluating the quality of service or the performance of cars, the decision makers often use "*excellent*", "*good*" and "*poor*". Therefore, the linguistic multi-attribute decision making problems are very important and are being studied extensively. There are many decision making techniques to deal multi-attribute decision making problems with linguistic information [5]–[17]. The current major techniques can be mentioned: the direct computation techniques with words, the techniques which compute with fuzzy numbers that semantic support of language labels, the techniques involved in computing on indexes of linguistic terms, the techniques based on fuzzy linguistic representation model. The methods which compute with words directly can not only avoid the loss of any linguistic information, but also simple and very convenient in calculation [3]. Yang Xu *et al.* have proposed a model that uses linguistic truth-valued logic to solve decision making problems. Following the idea of Computing with Words methodology, Shuwei Chen *et al.* proposed a logical reasoning framework based on a lattice ordered linguistic truth-valued logic for linguistic multi-criteria decision making problems [18]. Shuwei Chen *et al.* used a Linguistic lattice implication algebra struct with 9 modifiers and 2 generators for modeling the linguistic terms. That is, they used a pair of one linguistic modifier and one generator to represent a linguistic value. In realty, however, there are decision making situations where linguistic information needs to be represented by string of linguistic modifiers with length greater than one. For example, a car evaluation might be stated as follows: "*The car is very very cheap*" or "*The car is cheap*" is "*very very true*".

Nguyen and Wechler introduced the theory of hedge algebras [19]. These theories are an algebraic approach to linguistic modifiers in Zadeh's fuzzy logic. The types of hedge algebras and methods of computing with words have been developed in [20]–[23]. Le and Tran have proposed L-Mono-HA as a form of linear hedge algebra and limited to the length of the string of hedges [22]. In [22], the authors proposed linguistic many-valued logic by extending many-valued logic and choosing Łukasiewicz operators $\vee$, $\wedge$, $\otimes$, $\oplus, \neg, \rightarrow$for linguistic-valued domain, which represent linguistic information. Following the Hedge algebra topology and the linguistic many-valued logic, we propose a method, where Hedge algebra is used to form linguistic values involved in the MADM problems, and linguistic many-valued logic is used to deduce the final decision making outcome. This study is conducted with the view to apply linguistic valued qualitative aggregation and reasoning method to address MADM problems with linguistic information. In our method, linguistic terms are directly handled without numerical approximation, which takes advantage of linguistic information processing and shows the benefit of Hedge algebra. The rest of this paper is structured as follows. Section 2 introduces the linguistic many-valued logic for representing the linguistic information involved in decision making problems. Section 3 introduces linguistic reasoning-based our proposed method for dealing linguistic multi-criteria decision making problems. Section 4 presents an illustration example for our proposal method. Section 5 is the conclusion

2. Linguistic Many-Valued Logic for Representing Linguistic Information

The hedge algebrastheory informed by Nguyen and Wechler in [19] is an algebraic approach to linguistic hedges in Zadeh's fuzzy logic. Symmetric hedge algebras and linguistic-valued logic was introduced by Nguyen and Tran in [24]. Concepts, properties of the monotonous hedge algebra that have been researched in [19]–[23], [25]–[27].

2.1. Hedge Algebra

Let $(X, G, H, \leq)$ be an abstract algebra where X is the term set, G is the set of generators, H is the set of hedge operators or modifiers, and $\leq$ is partial order on X. Each hedge operator (hedge in short) $h \in H$ can be regarded as a unary function h: X → X; x ↦ hx. Moreover, suppose that each hedge is an ordering function, i.e., $\forall h \in H, \forall x \in X$: $hx > x$ or $hx < x$. Let I ∉ Hbe the identity hedge, i.e., $Ix = x$ for all $x \in X$. The properties of hedges are described as follow.

Definition 1: Given two hedges h, k, we say that:

- h and k are converse if $x \in X$: $hx > x$ iff $kx < x$;
- h and k are compatible if $x \in X$: $hx > x$ iff $kx > x$;
- $h \geq k$, if $x \in X$: $(hx \leq kx \leq x)$ or $(hx \geq kx \geq x)$;
- $h > k$ if $h \geq k$ and $h \neq k$;
- h is positive w.r.t. k if $x \in X$: $(hkx < kx < x)$ or $(hkx > kx > x)$;
- $h$is negative w.r.t. k if $x \in X$: $(kx < hkx < x)$ or $(kx > hkx > x)$.

Definition 2 *(Hedge algebra):* An abstract algebra $(X, G, H, \leq)$, with H decomposed into $H^+$and H^-as above, is called a hedge algebra (HA, for short) if it satisfies the followingproperties:

- Each hedge operation is either positive or negative w.r.t. the others, including itself.

- If terms u and v are independent, i.e., $u \notin H(v)$ and $v \notin H(u)$, then for all $x \in H(u)$, we have $x \notin H(v)$. In addition, if u and v are incomparable (i.e., $u \nless v$and $v \nless u$), then so are x and y, for every $x \in H(u)$ and $y \in H(v)$.
- If $x \neq hx$ then $x \notin H(hx)$ and if $h \neq k$and $hx \leq kx$ then $h'hx \leq k'kx$, for all h, k, h' and k' in H. Moreover, if $hx \neq kx$ then hx and kx are independent.
- If $u \notin H(v)$ and $u \leq v$(resp., $u \geq v$) then $u \leq hv$ (resp., $u \geq hv$), for any hedge h.

Among HAs, symmetric ones are the most frequently used, in which there are exactly two generators, like e.g., $G = \{dissatisfy, satisfy\}$ or $\{false, true\}$, etc.In this paper, we only consider symmetric HA. Let $G = \{c^-, c^+\}$, where $c^- < c^+$, they are called positive and negative generators respectively.

Definition 3 *(Linear symmetric HA)*: An abstract algebra $(X, G, H, \leq)$ is called a linear symmetric HA if it satisfies the following conditions:

- For all $h \in H^+$and $k \in H^-$, $h$and $k$are converse.
- The sets $H^+ \cup \{I\}$ and $H^- \cup \{I\}$are linearly ordered with the least element I.
- For each pair $h, k \in H$, either h is positive or negative w.r.t.k.
- If $h \neq k$ and $hx < kx$ then $h'hx < k'kx$, for all $h, k, h', k' \in H$ and $x \in X$.
- If $u \in H(v)$ and u $< v$ (u $> v$) then u $< hv$ ($u > hv$), respectively, for any $h \in H$.

Example 1: Consider a HA $(X, \{T, F\}, H, \leq)$, where $H = \{very, more, probably, mol\}$, "$T$" is "true", "$F$" is "false", "$mol$" is "more" or "less". H is decomposed into $H^+ = \{very, more\}$ and $H^- = \{probably, mol\}$. In H+∪{I} we have $very > more > I$, whereas in $H^- \cup \{I\}$ we have $mol > probably > I$.

- $very$ and $more$ are positive w.r.t. $very$ and $more$, negative w.r.t. $probably$ and mol;
- $probably$ and mol are negative w.r.t. $very$ and $more$, positive w.r.t. $probably$ and mol.

2.2. Monotonous Hedge Algebra

Definition 4 *(Monotonous hedge algebra):* A linear symmetric HA$(X, G, H, \leq)$ is called monotonic, denoted Mono-HA, if each $h \in H$ is positive w.r.t. all $k \in H^+(H^-)$, and negative w.r.t. all $h \in H^-(H^+)$. As defined, both sets $H^+ \cup \{I\}$ and $H^- \cup \{I\}$ are linearly ordered, however, $H = H^+ \cup H^- \cup \{I\}$ is not. For example, $very \in H^+$ and $mol \in H^-$ are not comparable. Let us extend the order relation on $H^+ \cup \{I\}$ and $H^- \cup \{I\}$ to one on $H \cup \{I\}$ as follows.

Definition 5: Given $h, k \in H \cup \{I\}$iff

- $h \in H^+, k \in H^-$; or
- $h, k \in H + \cup \{I\}$ and $h \geq k$; or
- $h, k \in H - \cup \{I\}$ and $h \geq k$, $h >_H k$ iff $h \geq_H k$ and $h \neq k$.

Example 2:

The order relation "$>_H$" in $H \cup \{I\}$, is$very >_H more >_H I >_H probably >_H mol$.

In [26], the authors show that, hedges of Mono-HA are "context-free", i.e., a hedge adjusts the meaning of a linguistic value independently of prior hedges in the string of hedge.

2.3. Finite Monotonous Hedge Algebra

In practice, humans only use linguistic values with a finite length of modifier for the vague concepts, i.e., humans only use a finite string of hedges for truth values [26]. This leads to the necessity to limit the hedge string's length in the truth value domain to make it not surpass L-any positive number [26]. Based on Mono-HA, we set finite monotonous hedge algebra to make linguistic truth value domain.

Definition 6 *(L-Mono-HA):* L-Mono-HA, L is a natural number, is a Mono-HA with standard presentation of all elements having the length not exceed $L+1$.

Definition 7 *(Linguistic truth-valued domain):* A linguistic truth-valued domain AX taken from a L-Mono-HA = $(X; \{c+, c-\}; H; \leq)$ is defined as AX $= X \cup \{0, w, 1\}$ of which 0, w, 1 are the smallest, neutral, and biggest elements respectively in AX.

Example 3: Given finite monotonous hedge algebra L-Mono-HA= $(X, \{c+, c-\}, H, \leq)$, $H = \{v = very, m = more, p = possibly\}$. We have the linguistic truth-valued domain AX = {0, vvc^-, mvc^-, vc^-, pvc^-, vmc^-, mmc^-, mc^-, pmc^-, c^-, vpc^-, mpc^-, pc^-, ppc^-, w, ppc^+, pc^+, mpc^+, vpc^+, c^+, pmc^+,mc^+, mmc^+, vmc^+, pvc^+, vc^+, mvc^+, vvc^+, 1}.

As stated in the definition of linear order relation in monotonous hedge algebra Mono-HA, elements in AX follow a linear order.

2.4. Linguistic Many-Valued Logic

In logic, the domain of truth values is represented by an algebraic structure with logical operations ∧, ∨, ¬, and →. Many-valued logic has a finite set of truth values consisting of linearly arranged elements on [0,1] and Łukasiewicz algebra is the algebraic structure for this truth-value domain.

Definition 8 *(Łukasiewicz algebra):* The structure ([0,1],∨, ∧, ⊗, ⊕,¬,→,0,1) is called Łukasiewicz algebra on the domain [0,1] with values in the range [0,1] and operators∨, ∧, ⊗, ⊕, ¬, →are defined as follows:

- $a \vee b = max(a, b)$;
- $a \wedge b = min(a, b)$;
- $a \otimes b = max(0, a + b - 1)$;
- $a \oplus b = min(1, a + b)$;
- $\neg a = 1 - a$;
- $a \to b = min(1, 1 - a + b)$.

According to Definition 7, the linguistic truth-valued domain AX = $\{v_i | i = 1,2,\dots,n\}$ with $v_1 = 0, v_n = 1$ in finite monotonous hedge algebra and linear order or $v_i \leq v_j \Leftrightarrow i \leq j$ ($v_i, v_j \in$AX). With the extension of the [0,1] value domain for the AX linguistic truth-valued domain. Then, based on Definition 8 we have the following definition:

Definition 9: Let $\mathcal{L}_n = (AX, \vee, \wedge, \otimes, \oplus, \neg, \to, 0, 1)$, and operators ∨,∧,⊗,⊕,¬,→ are defined as follows, for every $v_i, v_j \in AX$:

- $v_i \vee v_j = v_{max(i,j)}$;
- $v_i \wedge v_j = v_{min(i,j)}$;
- $v_i \otimes v_j = v_1 \vee v_{i+j-n}$.
- $v_i \oplus v_j = v_n \wedge v_{i+j}$;
- $\neg v_i = v_{n-i+1}$;
- $v_i \to v_j = v_{min(n, n-i+j)}$;

Then, based on the operations in the Łukasiewicz linguistic-valued algebra, we can build linguistic many-valued logic.

3. Decision making based on linguistic reasoning

In real-life, there are many situations, such as evaluating university faculty for tenure and promotion and evaluating the performance of different kinds of stocks and bonds, in which the information cannot be assessed precisely in numerical values but may be in linguistic variables. That is, variables whose values are not numbers but words or sentences in a natural or artificial language. For example, when evaluating the "teaching" or "research" of a university, linguistic values like "good", "fair" and "poor" are usually be used, and evaluating a car's speed, linguistic values like "very fast", "fast" and "slow" can be used. In this section, we propose a linguistic multi-attribute decision making approach using linguistic reasoning. The linguistic reasoning-based approach can infer a reasonable comprehensive evaluation based on the provided evaluations. We then introduce a suitable aggregation mechanism to derive a reasonably comprehensive assessment from the evaluations provided by the experts.

3.1. Linguistic Multi-Attribute Decision Making Problem

The linguistic multi-attribute decision making problem based on the Hedge algebra 2-Mono-HA, can now be described as follows.

Suppose that there are a finite set of alternatives, $X = \{x_1, x_2, \ldots, x_m\}$ $(m \geq 1)$, which are under evaluation according to several attributes or criteria, $U = \{u_1, u_2, \ldots, u_n\}$ associated with a set of weights, $W = \{w_1, w_2, \ldots, w_n\}$ $(n \geq 1)$. The expert-based evaluation terms to evaluate the substitutes are linguistic terms taken from the set AX, which is generated from 2-Mono-HA, and the weights,$W = \{w_1, w_2, \ldots, w_m\}$, which can be considered as the degree of importance of different attributes, and the value of weights may also be linguistic terms in the set AX.

The resolution for the linguistic multi-attribute decision making problem is to find a suitable aggregation or reasoning mechanism for reaching a reasonable comprehensive evaluation based on provided evaluations from the experts. For example, when evaluating several cars according to four attributes (criteria): "safety" (u_1), "price" (u_2), "comfort" (u_3) and "fuel economy" (u_4), one may express his opinion about one of these cars as "*Its safety is very very satisfied, price is rather satisfied, comfort is very quite satisfied, and fuel economy is rather dissatisfied*".

3.2. Building a Hedge Algebra Structure for the Elements of Decision Making Problem

As introduced in Subsection 3.1, the components of the linguistic multi-attribute decision making problem consist of the alternatives, the attributes (criteria), the weights associated with the attributes, and the rule base. Before applying the linguistic reasoning method based on linguistic many-valued logic, let us present our proposal for modeling of the linguistic multi-attribute decision making as follow:

- An attribute can be represented by a vague concept, e.g., "safety", "price", "comfort", "fuel economy", etc. A primitive proposition is a statement of the form, where is a variable, $p$is the association relationship between the variable $x$and the vague concept u. In this study, we regard $p$as the"satisfied" relationship.
- For example, the proposition "*The C car is fuel economy*" is represented as **satisfied**(C, "fuel economy"). In addition, in the decision making problem, x is an alternative. The composite propositions, which are composed by the primitive propositions with logical connectives ∧(and), ∨(or), ¬(not), and →(if-then), can be used for modeling more complex judgments.
- An evaluation for a primitive proposition is a mapping $fm: S \to AX$, $p(x,u) \longmapsto v$; where S is the set of primitive propositions, AX is the linguistic truth-value domain generated from 2-Mono-HA. Accordingly, we denote an assertion as $(p(x,u), v)$.
- For example, the evaluation of the sentence ("The C car is fuel economy" is "very true") is represented as (**satisfied**(C, "fuel economy"), "*very true*"); the evaluation of the sentence ("The C car is not fuel economy" is "rather true") is represented as (**satisfied**(C, "fuel economy"), "*rather false*").
- The value of weight can be taken from the set of linguistic truth values AX.

- The rule base is the rule used for the decision maker's judgment. It is used to create a comprehensive evaluation. We form the rule base as follows:

$$\text{if } p(x,u_1) \text{ is } r_1 \text{ and ... and } p(x,u_n) \text{ is } r_n \text{ then } q(x,c) \text{ is } e \tag{1}$$

or as follows:

$$\text{if } p(x,u_1) \text{ is } r_1 \text{ or ... or } p(x,u_n) \text{ is } r_n \text{ then } q(x,c) \text{ is } e \tag{2}$$

where n is number of attribute, $p(x,u_j)(j = 1..n)$, $q(x,c)$ are primitive propositions; p represents the "satisfied" relationship, q represents the "is" relationship between $x$and conclusion c (c_j can be "satisfied"); r_j is evaluation for $p(x,u_j)$ $(j = 1..n)$; e is the evaluation for $q(x,c)$.

For example, the sentence "If a car is very very cheap and very fuel economy, then the car is satisfied" can be represented as "if satisfied(x, "price") is very very true and satisfied(x, "fuel economy") is very true then is (x, "satisfied") is true". From the above description, we construct the decision matrix R as the following form:

$$\begin{array}{c|cccc|} & u_1 & u_2 & \cdots & u_n \\ \hline x_1 & v_{1,1} & v_{1,2} & \cdots & v_{1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_m & v_{m,1} & v_{m,2} & \cdots & v_{m,n} \\ \hline \end{array}$$

where $u_j$is the vague concept corresponding to the jth attribute, x_i is the ith alternative, $v_{i,j}$is the truthdegree of the evaluation of $p(x_i,u_j)$.

Assume that the decision maker needs to perform a comprehension evaluation of the alternatives using an "if-then" inference, for example: "*If a car is very very cheap and very fuel economy, then the car is satisfied*", or "*If a car is very very cheap or very fuel economy, then the car is very satisfied*". Thus, the linguistic multi-attribute decision making problem can be solved by linguistic reasoning method.

3.3. Decision Making based on Linguistic Reasoning

In this subsection, we shall present our proposal linguistic reasoning model for dealing the linguistic multi-attribute decision making problem. This model is based on the multi-conditional fuzzy model [25]. From the rule base, we have $n$inference rules corresponding to n attribute as follows, which are then used to infer the decision result.

$$\begin{gathered} \text{if } p(x,u_1) \text{ is } r_1 \text{then } q(x,c_1) \text{ is } e_1 \\ \text{if} p(x,u_2) \text{ is } r_2 \text{ then } q(x,c_2) \text{ is } e_2 \\ \cdots \\ \text{if } p(x,u_n) \text{ is } r_n \text{ then } q(x,c_n) \text{ is } e_n \end{gathered}$$

In addition, we see that if the evaluation result corresponding to attribute $u_j$is v_j, then the result e_j taking into account the weight w_j with respect to attribute u_j is $(w_j \otimes r_j)$; $j = 1,\ldots,n$. We define a simpler form as follows:

$$\begin{gathered} (p(x,u_1),r_1) \rightarrow (q(x,c_1),w_1 \otimes r_1) \\ (p(x,u_2),r_2) \rightarrow (q(x,c_2),w_2 \otimes r_2) \\ \cdots \\ (p(x,u_n),r_n) \rightarrow (q(x,c_n),w_n \otimes r_n) \end{gathered}$$

3.4. Linguistic Reasoning

In this subsection, we shall present the rules used for linguistic reasoning based on Hedge algebra. These rules have been proposed and studied in [19]–[21].

Definition 10: Let $\mathcal{P}$ be a set of propositional variables, $\mathcal{P} = \{P = p(x,u), Q = q(x,u'), F = f(x,u'), \dots\}$, and the operators $\wedge, \vee, \neg, \rightarrow$ are define as follow for all given hedge h and string of hedges σ, the following statements hold [28].

- $\neg(P,v) = (P,\neg v)$,
- $(P,\neg hv) = (P,h\neg v)$and
- $(P,\sigma\neg hv) = (P,\sigma h\neg v)$
- $P = P$and
- $\neg\neg P = P$;
- $P \vee Q = Q \vee P$and
- $P \wedge Q = Q \wedge P$;
- $F \vee (P \vee Q) = (F \vee P) \vee Q$and $F \wedge (P \wedge Q) = (F \wedge P) \wedge Q$;
- $F \wedge F = F$and
- $F \vee F = F$;
- $F \wedge (F \vee P) = F$and
- $F \vee (F \wedge P) = F$;
- $F \wedge (P \vee Q) = (F \wedge P) \vee (F \wedge Q)$and
- $F \vee (P \wedge Q) = (F \vee P) F \vee Q)$;
- $\neg(F \vee P) = (\neg F \wedge \neg P)$and $\neg(F \wedge P) = (\neg F \vee \neg P)$;
- $P \rightarrow Q = \neg P \vee Q$

3.4.1. Rule for hedge transfer:

Given h is hedge, δ is the string of hedges, the hedge moving rules are set:

$$(\mathrm{TR}_1): \frac{(p(x,hu),\delta c)}{(p(x,u),\delta hc)} \tag{3}$$

3.4.2. Generalized modus ponens

Given δ, σ, and δ′ are the hedge strings, the generalized modus ponens (GMP) was proposed [19]:

$$(\mathrm{GMP}): \frac{(p(x,u) \rightarrow q(y,v),\delta c),(p(x,u),\sigma c)}{(q(y,v),\delta c \otimes \sigma c)} \tag{4}$$

$$(\mathrm{EGMP}): \frac{((p(x,u),\delta c) \rightarrow (q(y,v),\sigma c),(p(x,u),\delta' c))}{((Q,\delta' c \otimes (\delta c \rightarrow \sigma c)))} \tag{5}$$

EGMP is an extension of GMP;

$$(\mathrm{NGMP}): \frac{((p(x,\neg u) \rightarrow q(y,v),v_i),(p(x,u),v_j))}{((q(y,v),v_i \otimes \neg v_j))} \tag{6}$$

$$\mathrm{ENGMP}: \frac{((p(x,\neg u),v_i) \rightarrow (q(y,v),v_j),(\neg q(x,u),v_k))}{((q(y,v),(v_i \rightarrow v_j) \otimes \neg v_k))} \tag{7}$$

ENGMP is an extension of NGMP

3.4.3. Generalized modus ponens with linguistic modifiers

The rules of generalized Modus ponens with linguistic modifiers (GMPLM) have been introduced in [21], [26]. Given α, β, δ, σ, θ, ∂, α', β', δ', θ', and ∂' are the hedge strings; get $\alpha = h_1 h_2 \dots h_k$, symbol$\alpha^{-1} = h_k \dots h_2 h_1$, the rules are:

$$\text{GMPLM: } \frac{(p(x,\delta u) \rightarrow q(y,\partial v),\alpha c),(p(x,\delta' u),\alpha' c)}{(q(y,\partial v),\alpha c \otimes \delta^{-1}(\alpha' \delta'^{-1} c))} \quad (8)$$

$$\text{EGMPLM:} \frac{(p(x,\delta u),\alpha c) \rightarrow (q(y,\partial v),\beta c),(p(x,\delta' u),\alpha' c)}{(q(y,\partial' v),(\alpha\delta^{-1} c \rightarrow \beta\,\partial^{-1} c) \otimes (\alpha 0 \delta'^{-1} c))} \quad (9)$$

$$\text{NGMPLM:} \frac{(p(x,\neg(\delta u)) \rightarrow q(y,\partial v),\alpha c),(p(x,\delta' u),\alpha' c)}{\left(q(y,\partial v),\alpha c \otimes \neg(\delta^{-1}(\alpha' \delta'^{-1} c))\right)} \quad (10)$$

$$\text{ENGMPLM:} \frac{(p(x,\delta u),\alpha c) \rightarrow (q(y,\partial v),\beta c),(p(x,\delta' u),\alpha' c)}{(q(y,\partial' v),(\alpha\delta^{-1} c \rightarrow \beta\,\partial^{-1} c) \otimes \neg(\alpha' \delta'^{-1} c))} \quad (11)$$

The overall procedure of the linguistic multi-attribute decision making approach based on linguistic reasoning is summarized as follows.

Step 1: Construct the algebraic structure 2-Mono-HA for modeling the linguistic terms involved in the decision making problem;

Step 2: Construct the rule base using the RT1rule;

Step 3: Ask the decision-makers to provide evaluations about the alternatives, from that, we get the decision matrix R;

Step 4: Construct decision matrix $R^*$from the original decision matrix Rusing the linguistic reasoning rules that presented in Subsection 3.4;

Step 5: Elaborate the comprehensive evaluation by aggregating the n evaluations obtained in 3.4. In case, the rules that have a logical relationship are conjunction (and), then the final evaluation (comprehensive evaluation) is $e_i = \oplus_{j=1}^{n} r_{i,j}^{*}$(or $\vee_{j=1}^{n} r_{i,j}^{*}$). Otherwise, the logical relation is disjunction (or), then the final evaluation will be$e_i = \otimes_{j=1}^{n} r_{i,j}^{*}$ (or $\wedge_{j=1}^{n} r_{i,j}^{*}$);

Step 6: Rank the alternatives by their value of comprehensive evaluation if necessary.

4. Illustrative Example

In this section, we present an example of universities evaluation. We describe the problem as follows.

Let $H = \{v = very, m = more, p = possibly\}$, $G = \{c^+ = \text{“}true\text{”}, c^- = \text{“}false\text{”}\}$, we have the linguistic truth-valued domain, which is generated by 2-Mono-HA as follows: AX = $\{v_1 = 0, v_2 = vvc^-, v_3 = mvc^-, v_4 = vc^-, v_5 = pvc^-, v_6 = vmc^-, v_7 = mmc^-, v_8 = mc^-, v_9 = pmc^-, v_{10} = c^-, v_{11} = vpc^-, v_{12} = mpc^-, v_{13} = pc^-, v_{14} = ppc^-, v_{15} = w, v_{16} = ppc^+, v_{17} = pc^+, v_{18} = mpc^+, v_{19} = vpc^+, v_{20} = c^+, v_{21} = pmc^+, v_{22} = mc^+, v_{23} = mmc^+, v_{24} = vmc^+, v_{25} = pvc^+, v_{26} = vc^+, v_{27} = mvc^+, v_{28} = vvc^+, v_{29} = 1\}$.

Suppose that there are three universities to be evaluate (alternatives): x_1, x_2, and x_3. There are three attributes: “high quality” (u_1), “good research” (u_2), and “good service” (u_3). The weights associated with the three attributes are supposed to be $w_1 = v_{25}$, $w_2 = v_{22}$, and $w_3 = v_{24}$. The rule base is: “*If the university is very high quality, very more good research, and very good service the university is very satisfied*”.

The decision maker provides their preferences over the alternatives with respect to the attributes by using the additional evaluation in AX. We will construct the decision matrix R as Table 1.

Table 1. Universities Evaluation (Decision matrix R)

	u_1	u_2	u_3
x_1	v_{22}	v_{24}	v_{27}
x_2	v_{25}	v_{20}	v_{22}
x_3	v_{24}	v_{23}	v_{25}

For convenience, we illustrate the decision matrix R as a column chart. The column corresponds to the value of the decision matrix R, the column height indicate to the truth degree of the evaluation. Fig. 1 illustrates the decision matrix R given in Table 1.

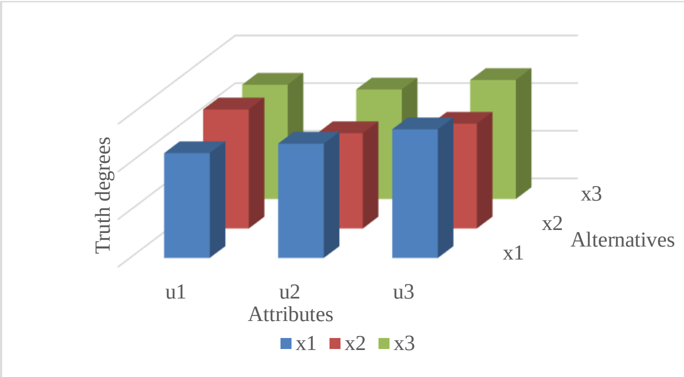

Fig. 1. Illustrative the decision matrix R given in Table 1. The linguistic truth value ofthe alternative x_1 on the attribute u_3 is the largest, and the linguistic truth value of the alternative x_2 on the attribute u_2 is the smallest.

For example, the value of $r_{1,1} = v_{22}$ in Table 1 expresses the evaluation of sentence "x_1*university is high quality*", i.e., "x_1*university is high quality*" is "*more true*".

The following steps show how to get an evaluation based on our method.

- Taking the weights into account, we get three inference rules from the rule base as follow.

$$(p(x, u_1), v_{26}) \rightarrow (q(x, c_1), v_{25} \otimes v_{26})$$
$$(p(x, u_2), v_{24}) \rightarrow (q(x, c_2), v_{22} \otimes v_{24})$$
$$(p(x, u_3), v_{26}) \rightarrow (q(x, c_3), v_{24} \otimes v_{26})$$

- Based on the rules described in Subsection 3.4, we can obtain the evaluations, as shown in Table 2.

Table 2. Universities Evaluation (Decision matrix R^*)

	u_1	u_2	u_3
x_1	v_{18}	v_{17}	v_{22}
x_2	v_{21}	v_{13}	v_{17}
x_3	v_{20}	v_{16}	v_{20}

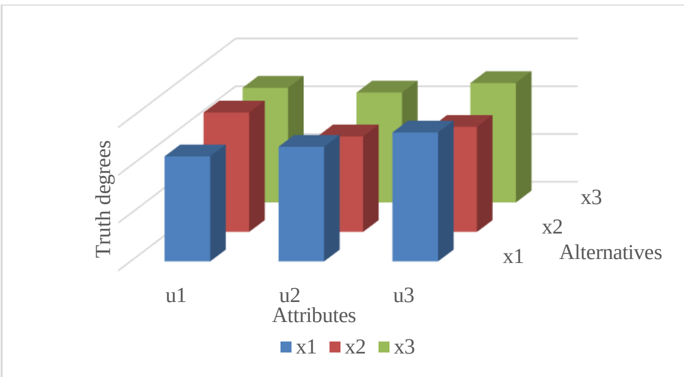

Fig. 2. Illustrative the decision matrix R^* given in Table 2.

In order to illustrate the reasoning process, we only take the top left one as an example, and the others can be done similarly. Denote P stands for $p(x, u_1)$ and Q stands for $q(x, c_1)$, the value of $r^*_{1,1}$ can be computed as follows:

- $r_{1,1} = v_{22}$
- apply EGMP rule: $\frac{(P,v_i)\rightarrow(Q,v_j),(P,v_k)}{(Q,v_k\otimes(v_i\rightarrow v_j))}$
- $r^*_{1,1} = \frac{(P,v_{26})\rightarrow(Q,v_{25}\otimes v_{26}=v_{22}),(P,v_{22})}{(Q,v_{22}\otimes(v_{26}\rightarrow v_{22}))} = v_{22} \otimes (v_{26} \rightarrow v_{22}) = v_{22} \otimes v_{25} = v_{18}$

From the decision matrix R^*, we can obtain the comprehensive evaluations with respect to different universities with the assumption that the logical relation between the attribute is disjunction (or), as $e_1 = v_{22}$, $e_2 = v_{21}$, and $e_3 = v_{20}$, which can be interpreted in natural language as "x_1 *university is more satisfied*", "x_2 *university is possibly more satisfied*", and "x_2 *university is satisfied*".

5. Conclusion

Multi-attribute decision making based on linguistic approach helps to solve the problem of decision making in the uncertain environment and the information cannot be assessed precisely in a quantitative form. The linguistic reasoning-based approximation process is a powerful approach for decision making problems. The theory of Hedge algebra and the linguistic many-valued logic are well suited for linguistic reasoning, and this paper demonstrated that these theories can be applied effectively to the linguistic multi-attribute decision making problems by our proposed method. Our method also provides a novel approach to linguistic decision making problems based on computing with words methodology. In future, we further study to expand our approach for group decision making problems in qualitative and uncertain environments.

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Declarations

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